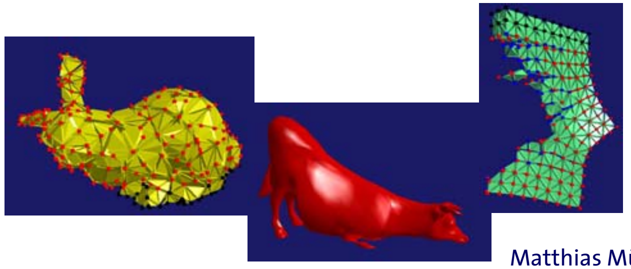
ETH Zurich

Interactive Simulation of Elasto-Plastic Materials using the Finite Element Method



Matthias Müller
Seminar – Wintersemester 02/03

Outline

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Mass-Spring vs. FEM

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1. Discretization of an object into **mass points**
2. Representation of forces between mass points with **springs**
3. Computation of the dynamics

➡ deformable mass-spring system

1. Discretization of an object into elements (tetrahedra)
2. Discretization of continuous energy equations into algebraic equations for forces acting at vertices
3. Computation of the dynamics

➡ deformable FEM system

Pros and Cons of FEM

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Pros:
- No individual spring constants needed (only 2 known material parameters E,ν)
- No inversion problems (inverted tetrahedra produce forces)
- Stress and strain tensors allow
  - fracture and
  - plasticity simulations

Cons:
- (Pre-)compute stiffness matrix
- Store stiffness matrix (3x3) per edge
- Store original **and** actual positions of vertices

Outline

ETH Zurich

FEM vs. Mass-Spring
Stiffness
- The Stiffness Matrix
- Static/Dynamic Deformation

Continuum Mechanics and FEM
- Strain and Stress Tensors
- Continuous PDE's
- FEM Discretization

Plasticity
- Plastic Strain
- Update Rules

Fracture
- Principal Stresses
- Crack Computation

One-dimensional Spring

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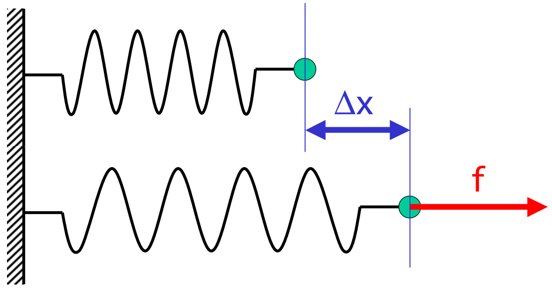

$$f = k \cdot \Delta x$$

Three-dimensional Object

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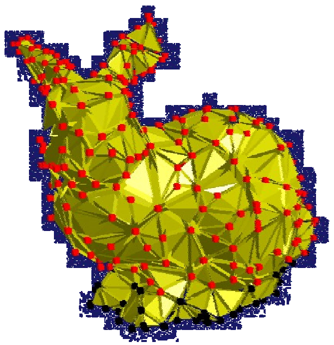
Finite Element Mesh
- 903 tetrahedra
- 393 vertices
- 3 x 393 = 1179 dof.

$$\Delta x = \begin{bmatrix} \Delta x_{1x} \\ \Delta x_{1y} \\ \Delta x_{1z} \\ \dots \\ \Delta x_{nx} \\ \Delta x_{ny} \\ \Delta x_{nz} \end{bmatrix} \quad f = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{1z} \\ \dots \\ f_{nx} \\ f_{ny} \\ f_{nz} \end{bmatrix}$$



Three-dimensional Object

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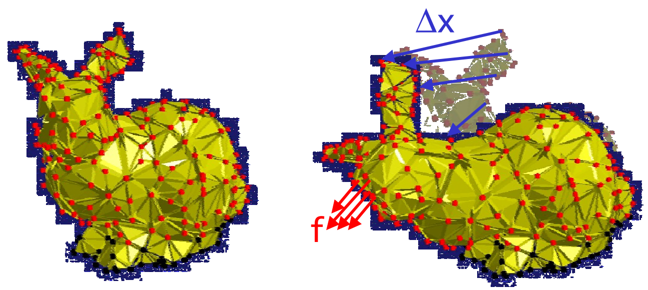

$f_{el} = K \cdot \Delta x$ (Stiffness Matix $K \in R^{3n \times 3n}$)

$f_{el} = F(\Delta x)$ (Function $F : R^{3n} \rightarrow R^{3n}$)

Static Deformation

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$f_{ext} = f_{el} = K \cdot \Delta x$

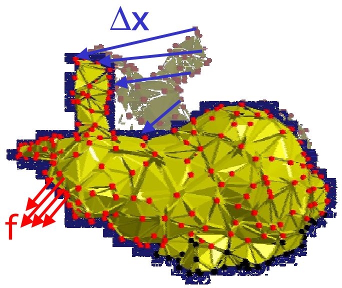
$\Delta x = K^{-1} \cdot f_{ext}$

Solve linear system
(Conjugate Gradients)

$f_{ext} = f_{el} = F(\Delta x)$

$\Delta x = F^{-1}(f_{ext})$

Solve non-linear system
(Newton-Raphson - generalized Newton-Method)



Dynamic Deformation

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$Mx'' + Cx' + K\Delta x = f_{ext}$

- Coupled system of 3n linear ODEs
- Explicit integration: No solver needed
- Implicit integration: Linear solver per time step

$Mx'' + Cx' + F(\Delta x) = f_{ext}$

- Coupled system of 3n non-linear ODEs
- Explicit integration: No solver needed
- Implicit integration: Linearize at every time step: K = dF/dx

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- FEM Discretization

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- Update Rules

Fracture
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- Crack Computation

Continuous Elasticity 1-d

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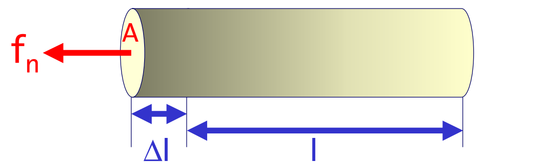

stress σ [N/m²]
(Normal-Spannung)

$f_n/A = E\, \Delta l/l$

strain ε [1]
(Dehnung)

Elasticity (Young's) Modulus
[N/m²]
Metal: ~10^{11} N/m²
Soft material: ~10^{6} N/m²

Continuous Elasticity 3-d

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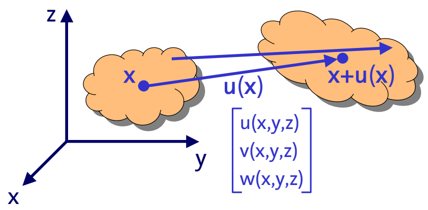
Deformation:



- Continuous 3-d vector field u: $R^3 \rightarrow R^3$
- Defined within undeformed object

Linear 3-d Strain

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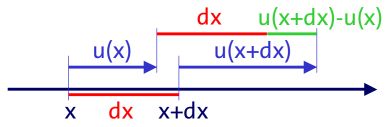
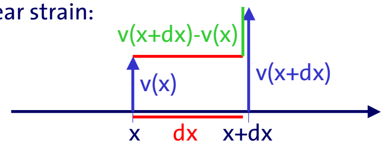
normal strain in x - direction:



$$\varepsilon_x = \frac{u(x+dx)-u(x)}{dx} = \partial/_{\partial x} u = u_x$$

shear strain:



$$\gamma_{xy} = \frac{v(x+dx)-v(x)}{dx} = \partial/_{\partial x} v = v_x$$

Linear 3-d Strain

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Linear strain tensor:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(x,y,z) = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} u_x & u_y + v_x & u_z + w_x \\ v_x + u_y & v_y & v_z + w_y \\ w_x + u_z & w_y + v_z & w_z \end{bmatrix}$$

Symmetric, 3x3 matrix → 6 vector:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Non-Linear 3-d Strain

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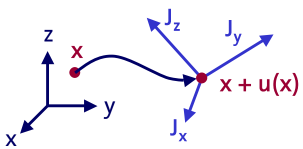
Green-Saint-Venant strain tensor:

- Transformation we use: $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{u}(\mathbf{x})$:
- Use Jacobian of transformation:

$$\mathbf{J} = (\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z) = \left(\begin{bmatrix} \partial/\partial x(x+u) \\ \partial/\partial x(y+v) \\ \partial/\partial x(z+w) \end{bmatrix}, \begin{bmatrix} \partial/\partial y(x+u) \\ \partial/\partial y(y+v) \\ \partial/\partial y(z+w) \end{bmatrix}, \begin{bmatrix} \partial/\partial z(x+u) \\ \partial/\partial z(y+v) \\ \partial/\partial z(z+w) \end{bmatrix} \right)$$

$$\boldsymbol{\varepsilon}_{\text{Green}} = \mathbf{J}^{\mathrm{T}}\mathbf{J} - \mathbf{I}$$

- Interpretation:



3-d Stress

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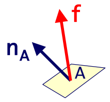
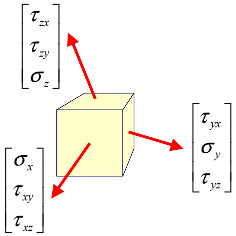
Stress is force per (oriented) area:

$$\boldsymbol{\sigma} = \frac{d\mathbf{f}}{d\mathbf{A}} = \frac{d\mathbf{f}}{dA \cdot \mathbf{n}_A}$$



$$\frac{d\mathbf{f}}{dA} = \boldsymbol{\sigma} \cdot \mathbf{n}_A = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \cdot \mathbf{n}_A$$



Constitutive Relation (isotropic)

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The stress tensor is symmetric, 3x3 matrix → 6 vector:

Hooke's law: $\boldsymbol{\sigma} = \mathbf{E}\,\boldsymbol{\varepsilon}$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} (1-\nu)c & \nu c & \nu c & & & \\ \nu c & (1-\nu)c & \nu c & & 0 & \\ \nu c & \nu c & (1-\nu)c & & & \\ & & & G & & \\ & 0 & & & G & \\ & & & & & G \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

$$c = \frac{E}{(1+\nu)(1-2\nu)}, G = \frac{E}{2(1+\nu)}$$

Only two scalar parameters:

E: Young's modulus, ν: Poisson ratio

Putting it all together

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Given **u**(**x**) we can compute

- strain ε(**x**) and
- stress **σ**(**x**) = E ε(**x**)

at every point **x** within the object.

→ Find **u**(**x**) such that corresponding stresses **σ**(**x**) are in balance with external forces **f**(**x**) everywhere within object:

$$\sigma_{x,x} + \tau_{xy,y} + \tau_{xz,z} + f_x = 0$$
$$\tau_{yx,x} + \sigma_{y,y} + \tau_{yz,z} + f_y = 0$$
$$\tau_{zx,x} + \tau_{zy,y} + \sigma_{z,z} + f_z = 0$$

- Strong formulation
- Coupled system of partial differential equations!

Energy Formulation

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- Energy U is a scalar
- U at point **x** is given by „displacement × force":

$$U_{\text{elastic}} = \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\,\boldsymbol{\varepsilon}$$

- The total Energy of the deformed body:

$$U_{\text{body}} = \int_{\text{body}} \left(\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\,\boldsymbol{\varepsilon} - \mathrm{u} \cdot \mathrm{f} \right) dV$$

- Given **f**, **E** we can compute U_{body} for any **u**(**x**)
- Find **u**(**x**) such that U_{body} is a minimum (δU = 0)

Outline

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Finite Element Formulation

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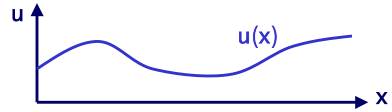
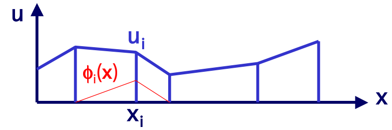
- So far we looked for a continuous field u(x)



- Now we look for $u_1, u_2, .. u_n$ at fixed locations: $x_1, x_2, .. x_n$
- and interpolate u(x) with fixed basis functions: $u(x) \approx \sum_i u_i \phi_i(x)$



Linear Displacement Tetrahedron

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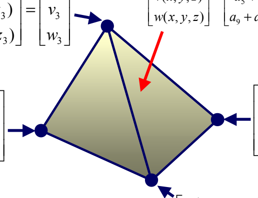
- 12 unknowns (a_1..a_{12}), 12 equations
- u_i, v_i, w_i are variables, x_i, y_i, z_i are given numbers

$$\begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix} = \begin{bmatrix} a_1 + a_2 x + a_3 y + a_4 z \\ a_5 + a_6 x + a_7 y + a_8 z \\ a_9 + a_{10} x + a_{11} y + a_{12} z \end{bmatrix}$$

$$\begin{bmatrix} u(x_3,y_3,z_3) \\ v(x_3,y_3,z_3) \\ w(x_3,y_3,z_3) \end{bmatrix} = \begin{bmatrix} u_3 \\ v_3 \\ w_3 \end{bmatrix}$$

$$\begin{bmatrix} u(x_1,y_1,z_1) \\ v(x_1,y_1,z_1) \\ w(x_1,y_1,z_1) \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix}$$



$$\begin{bmatrix} u(x_4,y_4,z_4) \\ v(x_4,y_4,z_4) \\ w(x_4,y_4,z_4) \end{bmatrix} = \begin{bmatrix} u_4 \\ v_4 \\ w_4 \end{bmatrix}$$

$$\begin{bmatrix} u(x_2,y_2,z_2) \\ v(x_2,y_2,z_2) \\ w(x_2,y_2,z_2) \end{bmatrix} = \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix}$$

Displacements

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The displacement function **u(x)** can be expressed as

- a matrix of basis functions **H(x)** times
- a vector of displacements:

$$\mathbf{u(x)} = \mathbf{H(x)} \cdot \hat{\mathbf{u}}$$

$$\begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{bmatrix}$$

Strain

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Linear displacements yield constant strain:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \mathbf{H}(\mathbf{x}) \cdot \tilde{\mathbf{u}} = \mathbf{B} \cdot \tilde{\mathbf{u}}$$

- Matrix B $\in \mathbb{R}^{6\times12}$ is constant (independent of x,y,z)
- B depends on the original geometry of the tetrahedron only

Stress and Energy

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Stress as a function of the displacements:

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} = \mathbf{E}\mathbf{B} \cdot \tilde{\mathbf{u}}$$

Energy as a function of the displacements:

$$U_{\text{element}} = \int_{\text{element}} \left(\frac{1}{2} \boldsymbol{\varepsilon}^T \mathbf{E} \boldsymbol{\varepsilon} \right) dV$$

$$= \int_{\text{element}} \frac{1}{2} \tilde{\mathbf{u}}^T \mathbf{B}^T \mathbf{E} \mathbf{B} \, \tilde{\mathbf{u}} \, dV$$

$$= \frac{1}{2} \tilde{\mathbf{u}}^T \left[\int_{\text{element}} \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV \right] \tilde{\mathbf{u}} = \frac{1}{2} \tilde{\mathbf{u}}^T \left[V_{\text{element}} \, \mathbf{B}^T \mathbf{E} \mathbf{B} \right] \tilde{\mathbf{u}}$$

$$= \frac{1}{2} \tilde{\mathbf{u}}^T \mathbf{K} \tilde{\mathbf{u}}$$

Stiffness Matrix

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$$U_{\text{body}} = \frac{1}{2} \tilde{\mathbf{u}}^T \mathbf{K} \tilde{\mathbf{u}}$$

Forces are the derivatives of the energy
with respect to the degrees of freedom:

$$\frac{\partial U_{\text{body}}}{\partial \tilde{\mathbf{u}}} = \mathbf{K} \tilde{\mathbf{u}} = \mathbf{f}$$

The matrix K $\in \mathbb{R}^{12\times12}$ is the stiffness matrix of the element!

$$\mathbf{K} = \int_{\text{element}} \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV = V_{\text{element}} \, \mathbf{B}^T \mathbf{E} \mathbf{B}$$

Assembly of elements

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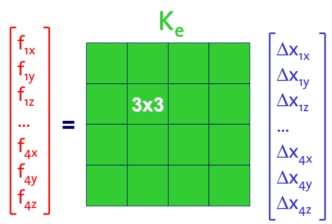
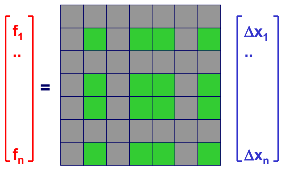
Single element:



Entire body:



Implementation

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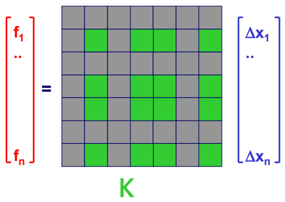

- K is sparse
- 3x3 block at (3i,3j) describes how $\Delta \mathbf{x}_j$ influences $\mathbf{f}_i$
- every vertex stores adjacency list of (3x3-matrix, vertex-reference) pairs

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Plastic Strain

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An element is under strain $\varepsilon_{\hat{u}}$ due to displacements $\hat{u}$:

$$\varepsilon_{\hat{u}} = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}]^T = B \cdot \hat{u}$$

A plastic element „stores“ strain in a state variable:

$$\varepsilon_{plastic}$$

The elastic strain (that causes internal forces) is now:

$$\varepsilon_{elastic} = \varepsilon_{\hat{u}} - \varepsilon_{plastic}$$

→ No internal forces are present when $\varepsilon_{\hat{u}} = \varepsilon_{plastic}$
Might be for $\hat{u} \neq 0$!

Plastic Update Rules

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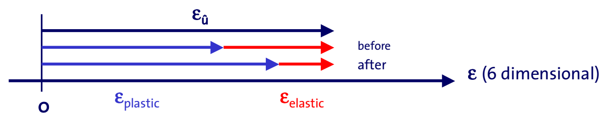
Initialization $\varepsilon_{plastic} = 0$

:

Update rule (every time step):

- Compute $\varepsilon_{\hat{u}} = B \cdot \hat{u}$ from actual displacements
- Compute $\varepsilon_{elastic} = \varepsilon_{\hat{u}} - \varepsilon_{plastic}$
- if $||\varepsilon_{elastic}|| >$ yield then $\varepsilon_{plastic} = \varepsilon_{plastic} +$ creep $\cdot\ \varepsilon_{elastic}$
- if $||\varepsilon_{plastic}|| >$ max then $\varepsilon_{plastic} = \varepsilon_{plastic} \cdot$ max / $||\varepsilon_{plastic}||$



Implementation

Since

$\varepsilon = B \cdot \hat{u}$

the displacements that correpsond to the plastic strain are:

$\hat{u}_{plastic} = B^{-1} \cdot \varepsilon_{plastic}$

and the correpsonding forces are:

$f_{plastic} = KB^{-1} \cdot \varepsilon_{plastic} = [vB^T EB]\, B^{-1} \cdot \varepsilon_{plastic} = vB^T E \cdot \varepsilon_{plastic}$

→ add plastic forces to external forces

Outline

Fracture criterion

- Break if internal elastic force exeeds threshold
- stress is a tensor
- the force w.r.t. normal $\mathbf{n}_A$ is:

$$\frac{d\mathbf{f}}{dA} = \boldsymbol{\sigma} \cdot \mathbf{n}_A = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \cdot \mathbf{n}_A$$

- Find $\mathbf{n}_{max}$ such that df/dA is maximal!
- $\mathbf{n}_{max}$ is direction of maximal tensile stress

Principal Stresses

- The stress tensor $\boldsymbol{\sigma}$ is symmetric
- → there is a rotation matrix R such that

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} = \mathbf{R}^T \begin{bmatrix} \sigma_x^p & 0 & 0 \\ 0 & \sigma_y^p & 0 \\ 0 & 0 & \sigma_z^p \end{bmatrix} \mathbf{R}$$

- the diagonal entries are the eigenvalues of $\boldsymbol{\sigma}$
- the columns of R are the corresponing eigenvectors
- there is always a rotated coordinate system where $\boldsymbol{\sigma}$ is diagonal!

Principal Stresses

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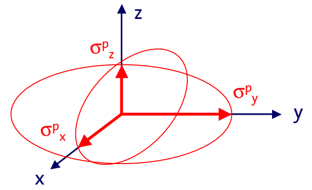

- the σ^p are the principal (extremal) stresses
- → find maximal eigenvalue of σ
- corresponding eigenvector is the direction of maximal stress

Crack computation

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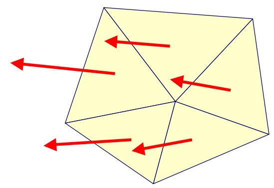
- for all elements: compute maximal tensile stress σ^p_{max}



Crack computation

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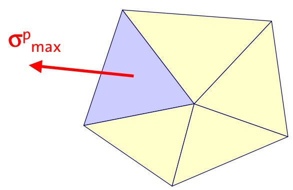
- for all elements: compute maximal tensile stress σ^p_{max}
- if σ^p_{max} exceeds yield stress



Crack computation

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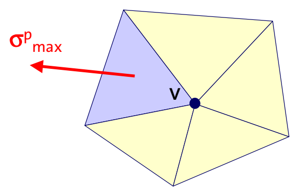
- for all elements: compute maximal tensile stress σ^p_{max}
- if σ^p_{max} exceeds yield stress
- select a vertex v (crack tip / random)



Crack computation

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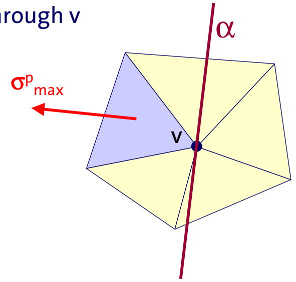
- for all elements: compute maximal tensile stress σ^p_{max}
- if σ^p_{max} exceeds yield stress
- select a vertex v (crack tip / random)
- set plane α normal σ^p_{max} to through v



Crack computation

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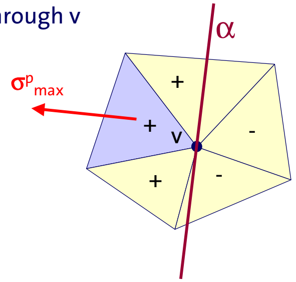
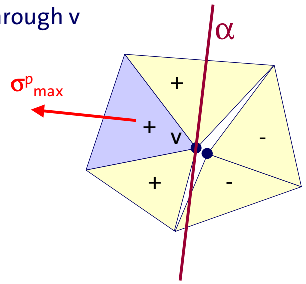
- for all elements: compute maximal tensile stress σ^p_{max}
- if σ^p_{max} exceeds yield stress
- select a vertex v (crack tip / random)
- set plane α normal σ^p_{max} to through v
- mark tetras w.r.t. α



Crack computation

ETH Zurich

- for all elements: compute maximal tensile stress σ^p_{max}
- if σ^p_{max} exceeds yield stress
- select a vertex v (crack tip / random)
- set plane α normal σ^p_{max} to through v
- mark tetras w.r.t. α
- split vertex



The End

ETH Zurich

Thank you for your attention!