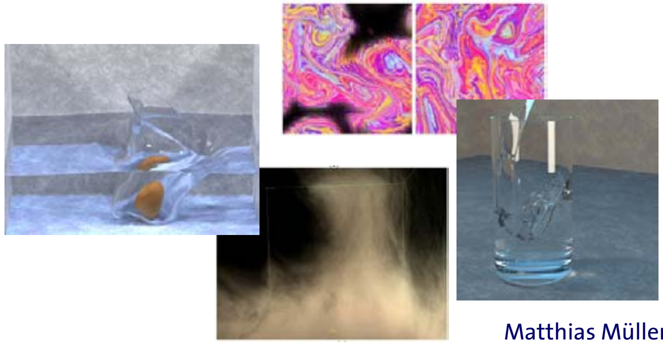
Fluid Equations

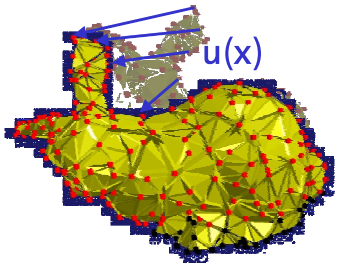


Matthias Müller
Seminar – Wintersemester 02/03

Outline

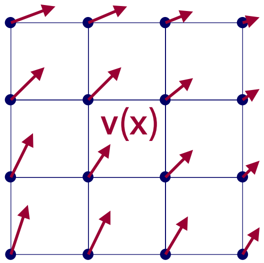
Lagrange vs. Euler

Lagrangian (displacement) approach



- displacement field u(x)
- follows particle
- we know where original particle is at any time

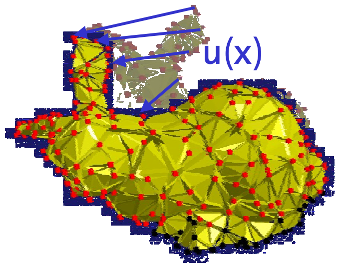
Eulerian (velocity) approach



- velocity field v(x)
- is fixed in space
- we don't know where original particle is

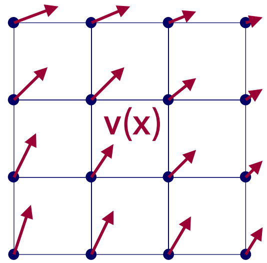
Lagrange vs. Euler

Lagrangian (displacement) approach



typically
- computed on a mesh
- Finite Element Method
- elastic objects

Eulerian (velocity) approach



typically
- computed on a (regular) grid
- Finite Differences Method
- fluids

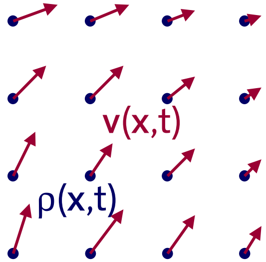
Two Continuous Quantities

ETH Zurich

Scalar field $\rho(\mathbf{x},t)$ [kg/m³]

Vector field $\mathbf{v}(\mathbf{x},t)$ [m/s]

$$\mathbf{v}(x,y,z,t) = \begin{bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{bmatrix}$$



For incompressible fluids: $\rho(\mathbf{x},t) \equiv 1$

Two Equations

ETH Zurich

At every point **x**:

Conservation of mass

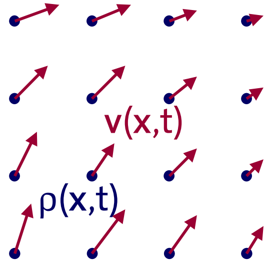
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

local increase | flow out

Conservation of momentum

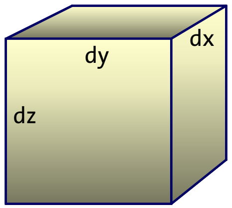
$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{v}$$

"m·a / volume" | "force / volume"



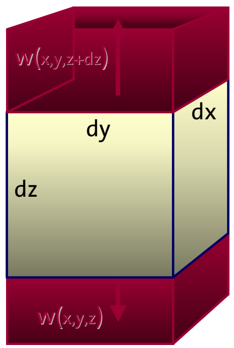
Density Flow Rate

ETH Zurich



Density Flow Rate

ETH Zurich

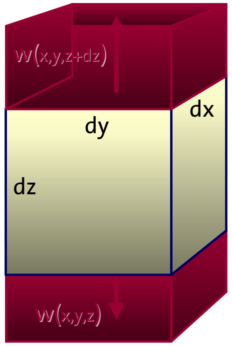


$w(x,y,z+dz) \cdot dx \cdot dy \cdot \rho(x,y,z+dz)$ mass flow rate out

$-w(x,y,z) \cdot dx \cdot dy \cdot \rho(x,y,z)$ mass flow rate out

Density Flow Rate

ETH Zurich



w(x,y,z+dz)·dx·dy ·ρ(x,y,z+dz) — mass flow rate out

net density flow rate out

= [w(x,y,z+dz)·dx·dy·ρ(x,y,z+dz)
 - w(x,y,z)·dx·dy·ρ(x,y,z)] /(dx·dy·dz)

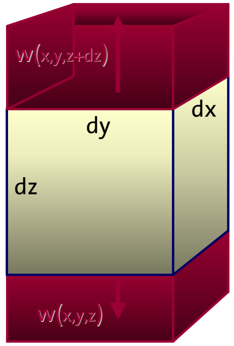
= [w(x,y,z+dz)·ρ(x,y,z+dz) -w(x,y,z)·ρ(x,y,z)] /dz

= ∂/∂z (w·ρ)

-w(x,y,z)·dx·dy ·ρ(x,y,z) — mass flow rate out

Total Density Flow Rate

ETH Zurich



total net density flow rate out

= ∂/∂x (u·ρ) + ∂/∂y (v·ρ) + ∂/∂z (w·ρ)

= ∇·(ρ**v**)

= div (ρ**v**)

density increase rate inside

= ∂ρ/∂t

mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

incompressible:

$$\nabla \cdot \mathbf{v} = 0$$

Momentum Conservation

ETH Zurich

Newton's second law per unit volume (Navier-Stokes, simplified!):

"m·a per volume" = "force per volume"

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{v}$$

Material derivative of velocity (following the fluid):

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial}{\partial t}\mathbf{v}(x(t), y(t), z(t), t) = \frac{\partial \mathbf{v}}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial \mathbf{v}}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial \mathbf{v}}{\partial z}\cdot\frac{\partial z}{\partial t} + \frac{\partial \mathbf{v}}{\partial t}\cdot\frac{\partial t}{\partial t}$$

$$= \frac{\partial \mathbf{v}}{\partial x}\cdot u + \frac{\partial \mathbf{v}}{\partial y}\cdot v + \frac{\partial \mathbf{v}}{\partial z}\cdot w + \frac{\partial \mathbf{v}}{\partial t}$$

$$= \mathbf{v}\cdot\nabla\mathbf{v} + \frac{\partial v}{\partial t}$$

External Forces

ETH Zurich

Newton's second law per unit volume:

"m·a per volume" = "force per volume"

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{v}$$

External forces:

- gravity force = mass · acceleration per volume
- other user applied forces (force per volume!)

Pressure

ETH Zurich

Newton's second law per unit volume:

"m·a per volume" = "force per volume"

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{v}$$

$$\boldsymbol{\sigma}_{\text{pressure}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

$$\mathbf{f}_{\text{int}} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} \sigma_{x,x} + \tau_{xy,y} + \tau_{xz,z} \\ \tau_{yx,x} + \sigma_{y,y} + \tau_{yz,z} \\ \tau_{zx,x} + \tau_{zy,y} + \sigma_{z,z} \end{bmatrix} = \begin{bmatrix} p_{,x} \\ p_{,y} \\ p_{,z} \end{bmatrix} = \nabla p$$

Pressure

ETH Zurich

Newton's second law per unit volume:

"m·a per volume" = "force per volume"

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{v}$$

What is p?

- for isotermal fluids:
- pV = const
- p = k/V = kρ

Viscosity

ETH Zurich

Newton's second law per unit volume:

"m·a per volume" = "force per volume"

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{v}$$

Simplified viscosity force for incompressible fluids:

$$\mu\nabla^2\mathbf{v} = \mu \begin{bmatrix} u_{,xx} + u_{,yy} + u_{,zz} \\ v_{,xx} + v_{,yy} + v_{,zz} \\ w_{,xx} + w_{,yy} + w_{,zz} \end{bmatrix}$$

Scalar μ is the fluid's viscosity

Putting it all together

ETH Zurich

So far we have:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -k\nabla\rho + \rho\mathbf{g} + \mu\nabla^2\mathbf{v}$$

Rearrange to get rates of change:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{v})$$

$$\frac{\partial\mathbf{v}}{\partial t} = -\mathbf{v}\cdot\nabla\mathbf{v} - \frac{k}{\rho}\nabla\rho + \mathbf{g} + \frac{\mu}{\rho}\nabla^2\mathbf{v}$$

Finite Difference Integration

ETH Zurich

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{k}{\rho} \nabla \rho + \mathbf{g} + \frac{\mu}{\rho} \nabla^2 \mathbf{v}$$

On a uniform grid using finite differences we get:

$$\frac{\partial \rho_{i,j,k}}{\partial t} = -\frac{\rho_{i,j,k} u_{i,j,k} - \rho_{i-1,j,k} u_{i-1,j,k}}{h} - \frac{\rho_{i,j,k} v_{i,j,k} - \rho_{i,j-1,k} v_{i,j-1,k}}{h} - \frac{\rho_{i,j,k} w_{i,j,k} - \rho_{i,j,k-1} v_{i,j,k-1}}{h}$$

Similar for change of velocity

(see "*A Fluid-Based Soft-Object Model*", IEEE Computer Graphics and Applications July 2002)

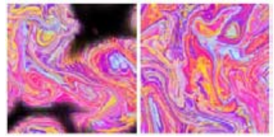
Time Integration

ETH Zurich

Time step using Euler Integration:

$$\rho_{i,j,k} = \rho_{i,j,k} + \Delta t \frac{\partial \rho_{i,j,k}}{\partial t}$$

$$\mathbf{v}_{i,j,k} = \mathbf{v}_{i,j,k} + \Delta t \frac{\partial \mathbf{v}_{i,j,k}}{\partial t}$$



Done! ☺

How to get a surface?

- let particles flow with velocity field
- particles define potential
- use levelset method