Surfaces from Point Samples

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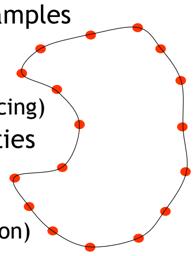
Motivation

- Many applications need definition of surface based on point samples
  - Reduction
  - Up-sampling
  - Interrogation (e.g. ray tracing)
- Desirable surface properties
  - Manifold
  - Smooth
  - Local (efficient computation)



Overview

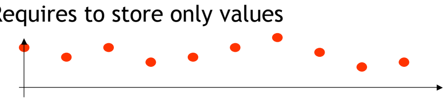
- Introduction & Basics
- Fitting Implicit Surfaces
- Projection-based Surfaces

Introduction & Basics

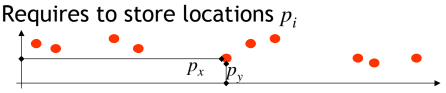
- Regular/Irregular
- Approximation/Interpolation
- Global/Local
- Standard techniques
  - LS, RBF, MLS
- Problems
  - Sharp edges, feature size/noise
- Functional/Manifold

Regular/Irregular

- Regular
  - Requires to store only values

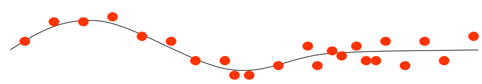


- Irregular
  - Requires to store locations p_i

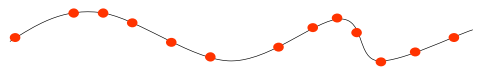


Approximation/Interpolation

- Noisy data -> Approximation

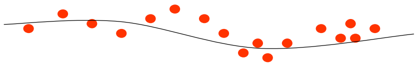


- Perfect data -> Interpolation

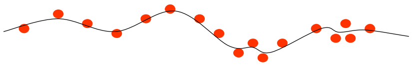


Global/Local

- Global approximation



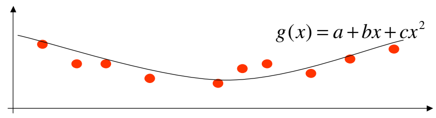
- Local approximation



- Locality comes at the expense of smoothness

Least Squares

- Fits a primitive to the data
- Minimizes squared distances between the p_i's and primitive g



$$\min_g \sum_i \left(p_{i_y} - g\left(p_{i_x}\right)\right)^2$$

Least Squares - Example

- Primitive is a polynomial

$$g(x) = \left(1, x, x^2, \ldots\right) \cdot \mathbf{c}^T$$

- $\min \sum_i \left(p_{i_y} - \left(1, p_{i_x}, p_{i_x}^2, \ldots\right)\mathbf{c}^T\right)^2 \Rightarrow$

$$0 = \sum_i 2 p_{i_x}^j \left(p_{i_y} - \left(1, p_{i_x}, p_{i_x}^2, \ldots\right)\mathbf{c}^T\right)$$

- Linear system of equations

Least Squares - Example

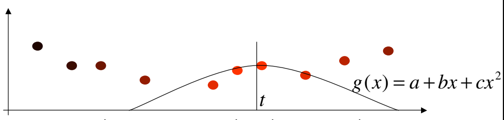
- Resulting system

$$0 = \sum_i 2 p_{i_x}^j \left(p_{i_y} - \left(1, p_{i_x}, p_{i_x}^2, \ldots\right)\mathbf{c}^T\right) \Leftrightarrow$$

$$\begin{pmatrix} 1 & x & x^2 & \cdots \\ x & x^2 & x^3 & \\ x^2 & x^3 & x^4 & \\ \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} y \\ yx \\ yx^2 \\ \vdots \end{pmatrix}$$

Moving Least Squares

- Compute a local LS approximation at t
- Weight data points based on distance to t



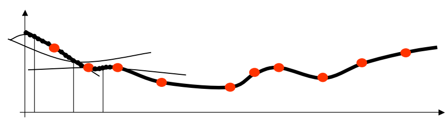
$$\min \sum_i \left(p_{i_y} - g\left(p_{i_x}\right)\right)^2 \theta\left(\left\| t - p_{i_x} \right\|\right)$$

Moving Least Squares

- The set

$$f(t) = g_t(t), g_t : \min_g \sum_i \left(p_{i_y} - g\left(p_{i_x}\right)\right)^2 \theta\left(\left\| t - p_{i_x} \right\|\right)$$

is a smooth curve, iff θ is smooth



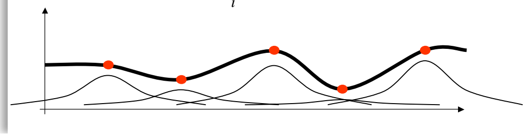
Moving Least Squares

- Typical choices for θ:
  - $\theta(d) = d^{-r}$
  - $\theta(d) = e^{-d^2/h^2}$
- Note: $\theta_i = \theta\left(\left\| t - p_{i_x} \right\|\right)$ is fixed
- For each t
  - Standard weighted LS problem
  - Linear iff corresponding LS is linear

Radial Basis Functions

- Represent interpolant as
  - Sum of radial functions r
  - Centered at the data points p_i

$$f(x) = \sum_i w_i r\left(\left\| p_i - x \right\|\right)$$



Radial Basis Functions

- Solve $p_{j_y} = \sum_i w_i r\left(\left\| p_{i_x} - p_{j_x} \right\|\right)$

  to compute weights w_i
- Linear system of equations

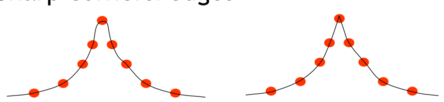
$$\begin{pmatrix} r(0) & r\left(\left\| p_{0_x} - p_{1_x} \right\|\right) & r\left(\left\| p_{0_x} - p_{2_x} \right\|\right) & \cdots \\ r\left(\left\| p_{1_x} - p_{0_x} \right\|\right) & r(0) & r\left(\left\| p_{1_x} - p_{2_x} \right\|\right) & \\ r\left(\left\| p_{2_x} - p_{0_x} \right\|\right) & r\left(\left\| p_{2_x} - p_{1_x} \right\|\right) & r(0) & \\ \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} p_{0_y} \\ p_{1_y} \\ p_{2_y} \\ \vdots \end{pmatrix}$$

Radial Basis Functions

- Solvability depends on radial function
- Several choices assure solvability
  - $r(d) = d^2 \log d$ (thin plate spline)
  - $r(d) = e^{-d^2/h^2}$ (Gaussian)
    - h is a data parameter
    - h reflects the feature size or anticipated spacing among points

Typical Problems

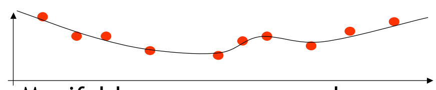
- Sharp corners/edges



- Noise vs. feature size

Functional/Manifold

- Standard techniques are applicable if data represents a function

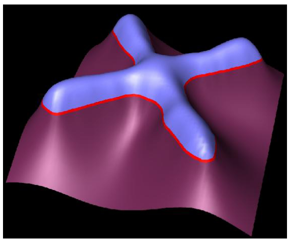


- Manifolds are more general

Implicits

- Each orientable n-manifold can be embedded in n+1 - space
- Idea: Represent n-manifold as zero-set of a scalar function in n+1 - space
  - Inside: $f(\mathbf{x})<0$
  - On the manifold: $f(\mathbf{x})=0$
  - Outside: $f(\mathbf{x})>0$

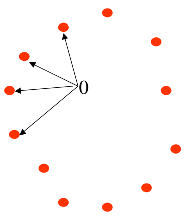
Implicits - Illustration



- Image courtesy Greg Turk

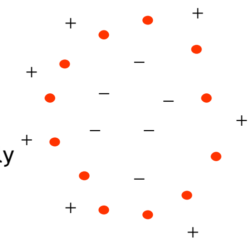
Implicits from point samples

- Function should be zero in data points
  - $f(p_i)=0$
- Use standard approximation techniques to find f
- Trivial solution: $f=0$
- Additional constraints are needed



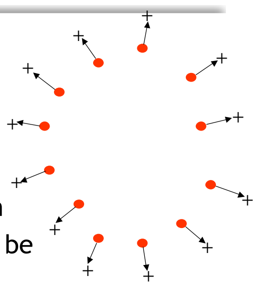
Implicits from point samples

- Constraints define inside and outside
- Simple approach (Turk, O'Brien)
  - Sprinkle additional information manually
  - Make additional information soft constraints



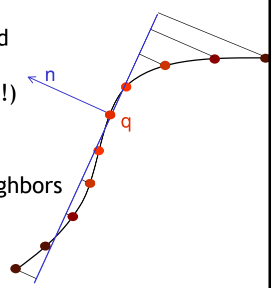
Implicits from point samples

- Use normal information as constraint
  - $f(p_i+n_i)=1$
- Normals could be computed from scan
- Or, normals have to be estimated



Estimating normals

- Two problems
  - Normal direction and
  - Orientation (Implicits are signed!)
- Normal direction by fitting a tangent
  - LS fit to nearest neighbors
  - Weighted LS fit
  - MLS fit

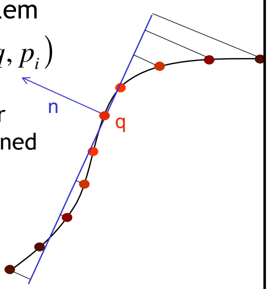


Estimating normals

- General fitting problem

$$\min_{\|n\|=1} \sum_i \langle q - p_i, n \rangle^2 \theta(q, p_i)$$

  - Problem is non-linear because n is constrained to unit sphere



Estimating normals

- The constrained minimization problem

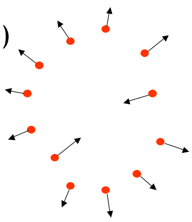
$$\min_{\|n\|=1} \sum_i \langle q - p_i, n \rangle^2 \theta_i$$

is solved by the eigenvector corresponding to the smallest eigenvalue of

$$\begin{pmatrix} \sum_i (q_x - p_{i_x})^2 \theta_i & \sum_i (q_x - p_{i_y})^2 \theta_i & \sum_i (q_x - p_{i_z})^2 \theta_i \\ \sum_i (q_y - p_{i_x})^2 \theta_i & \sum_i (q_y - p_{i_y})^2 \theta_i & \sum_i (q_y - p_{i_z})^2 \theta_i \\ \sum_i (q_z - p_{i_x})^2 \theta_i & \sum_i (q_z - p_{i_y})^2 \theta_i & \sum_i (q_z - p_{i_z})^2 \theta_i \end{pmatrix}$$

Estimating normals

- Consistent orientation
  - Problem is NP-hard
- Greedy approach (Hoppe)
  - Compute spanning tree based on graph of k-nearest neighbors
  - Orient consistently along spanning tree



Computing Implicits

- Given N points and normals p_i, n_i and constraints $f(p_i) = 0, f(p_i + n_i) = 1$
- Let $p_{i+N} = p_i + n_i$
- An RBF approximation

$$f(\mathbf{x}) = \sum_i w_i r(\|p_i - \mathbf{x}\|)$$

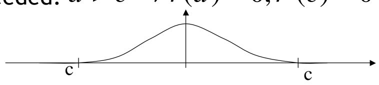
leads to $2N$ linear equations in $2N$ unknowns (a $2N \times 2N$ matrix)

Computing Implicits

- Practical problems: $N > 10000$
- Matrix solution becomes difficult
- Two solutions
  - Sparse matrices allow iterative solution
  - Smaller number of RBFs

Computing Implicits

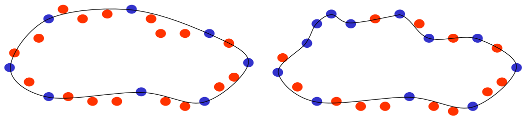
- Sparse matrices $\begin{pmatrix} r(0) & r(\|p_0 - p_1\|) & r(\|p_0 - p_2\|) & \cdots \\ r(\|p_1 - p_0\|) & r(0) & r(\|p_1 - p_2\|) & \\ r(\|p - p_0\|) & r(\|p_2 - p_1\|) & r(0) & \\ \vdots & & & \ddots \end{pmatrix}$
  - Needed: $d > c \rightarrow r(d) = 0, r'(c) = 0$



  - Compactly supported RBFs

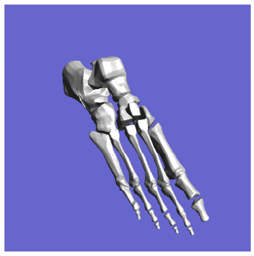
Computing Implicits

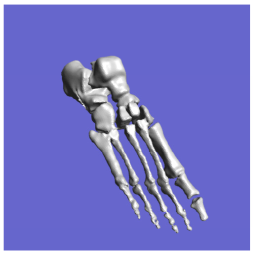
- Smaller number of RBFs
- Greedy approach (Carr et al.)
  - Start with random small subset
  - Add RBFs where approximation quality is not sufficient



RBF Implicits - Results

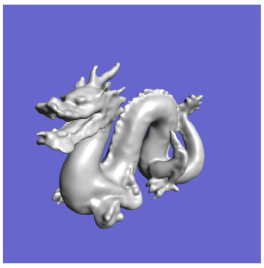
- Images courtesy Greg Turk





RBF Implicits - Results

- Images courtesy Greg Turk

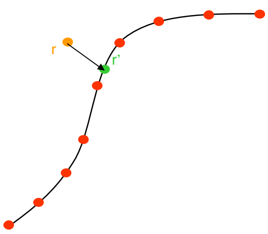


Implicits - Conclusions

- Scalar field is underconstrained
  - Constraints only define where the field is zero, not where it is non-zero
- Signed fields restrict surfaces to be unbounded
  - All implicit surfaces define solids

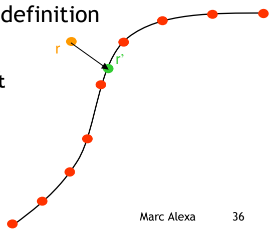
Projection

- Idea: Map space to surface
- Surface is defined as fixpoints of mapping



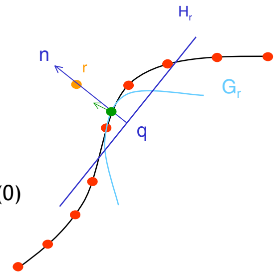
Surface definition

- Projection procedure (Levin)
  - Local polyonmial approximation
    - Inspired by differential geometry
  - "Implicit" surface definition
  - Infinitely smooth &
  - Manifold surface



Surface Definition

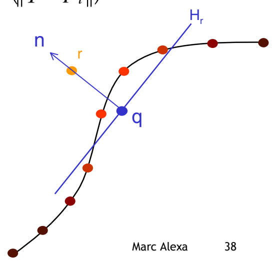
- Constructive definition
  - Input point r
  - Compute a local reference plane H_r=<q,n>
  - Compute a local polynomial over the plane G_r
  - Project point r'=G_r(0)
  - Estimate normal



Local Reference Plane

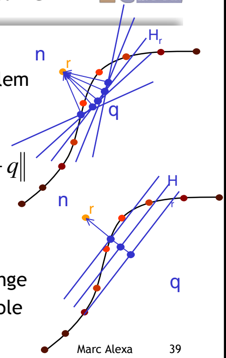
- Find plane $H_r = \langle q,n \rangle + D$
  - $\min_{q,\|n\|=1} \sum_i \langle q - p_i, n \rangle^2 \theta(\|q - p_i\|)$
  - $\theta(\mathrm{d}) = e^{d^2/h^2}$
    - h is feature size/ point spacing
  - H_r is independent of r's distance
  - Manifold property

Weight function based on distance to q, not r



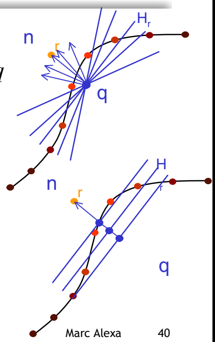
Local Reference Plane

- Computing reference plane
  - Non-linear optimization problem
- Minimize independent variables:
  - Over n for fixed distance $\|r - q\|$
  - Along n for fixed direction n
  - q changes -> the weights change
  - Only iterative solutions possible



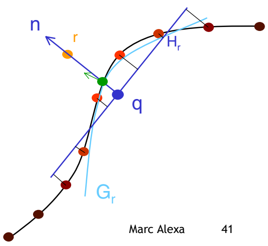
Local Reference Plane

- Practical computation
  - Minimize over n for fixed q
    - Eigenvalue problem
  - Translate q so that $r = q + \|r - q\| n$
    - Effectively changes $\|r - q\|$
  - Minimize along n for fixed direction n
    - Exploit partial derivative



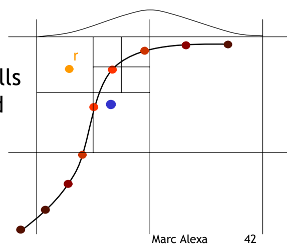
Projecting the Point

- MLS polyonomial over H_r
  - $\min_{G \in \Pi_d} \sum_i \left(\langle q - p_i, n \rangle - G\left(p_i|_{H_r}\right) \right)^2 \theta(\|q - p_i\|)$
  - LS problem
  - r'=G_r(0)
  - Estimate normal



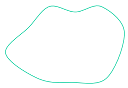
Spatial data structure

- Regular grid based on support of θ
  - Each point influences only 8 cells
- Each cell is an octree
  - Distant octree cells are approximated by one point in center of mass

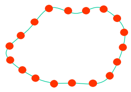


Error bounds

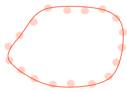
- Paradigm:
  - Given surface S



  - Point set $P = \{p_i\}$ sampled from S

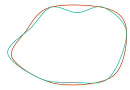


  - ($r_i \in S$) defines S_R



Error bounds

- Approximation error of S_P to S
  - MLS error approximating a function f with a polynomial g: $\|f - g\| \leq M \cdot h^{m+1}$
    - $M \in O\left(\left\| f^{(m+1)} \right\|\right)$
    - m = degree of polynomial
  - S_P is approximated by a polynomial in each point
  - $\|S - S_p\| \leq M \cdot h^{m+1}$



Error bounds

- Conclusions
  - Remark: Curvature is a useful criterion only for piecewise linear surfaces
  - Generally: Higher order derivatives are not accessible
  - Quality of representation is mainly dictated by h
  - Number of points control h
  - Increase/decrease number of points to adjust the quality of representation

Conclusions

- Projection-based surface definition
  - Surface is smooth and manifold
  - Surface may be bounded
  - Representation error mainly depends on point density
  - Adjustable feature size h allows to smooth out noise

Some References

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