T2 – Filtering, Fourier Transform, Aliasing

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Discussion P3

- b) IV: $\mathbf{n}' \bullet \mathbf{p}' = 0$

$$(\tilde{\mathrm{A}}\mathbf{n})^{\mathrm{T}} \cdot (\mathrm{A}\mathbf{p}) = \mathbf{n}^{\mathrm{T}}\tilde{\mathrm{A}}^{\mathrm{T}} \cdot \mathrm{A}\mathbf{p} = 0$$

- c) I: Einheitsquaternion!

$$\mathbf{n} = \frac{[3\ 0\ 4]}{\|[3\ 0\ 4]\|} = [0.6\ 0\ 0.8]$$

Aufgabe 1 – Convolution

- Convolution: combine two functions into one
- Definition:

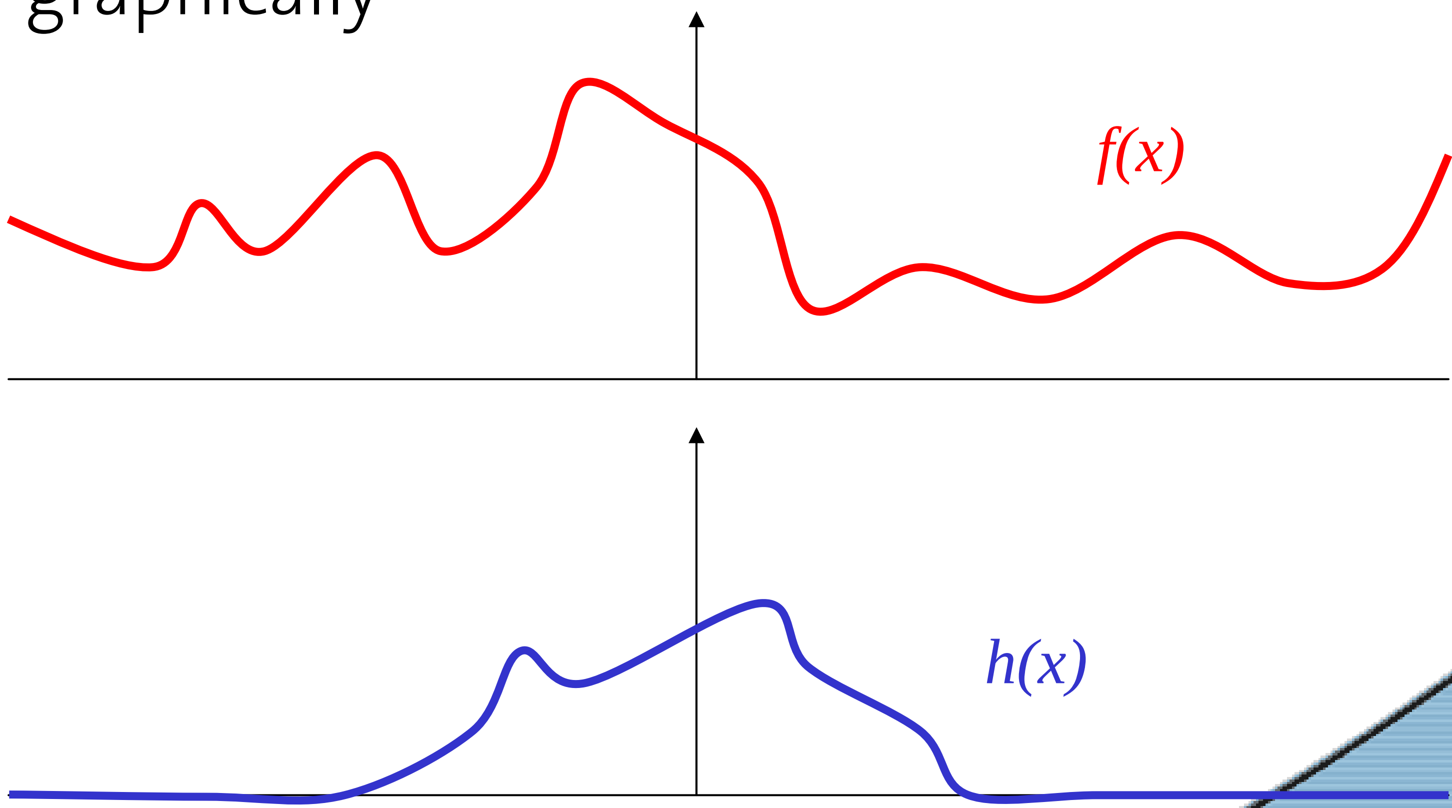
inversion

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x - \alpha) \, d\alpha$$

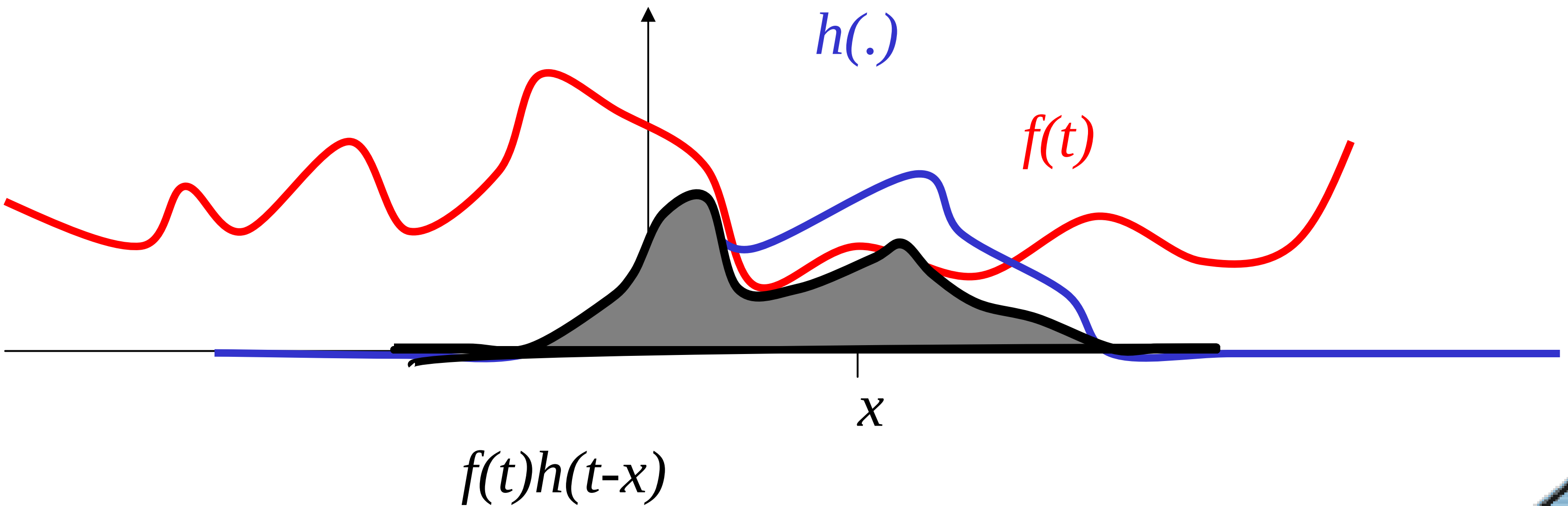
$$f(x) * g(x) = \sum_{m=0}^{M-1} f(m) \, g(x - m)$$

Convolution

- graphically



Convolution



Convolution

- Convolution animated:
  http://mathworld.wolfram.com/Convolution.html
- Aufgabe 1:
  - sequence of impulses – continuous formula, but only evaluates to values != 0 for impulses
    => scale with amplitude of impulse
  - can be solved using Maple, Matlab

Aufgabe 2 & 3

- Fourier Analysis
  - only basics here
  - more in lecture
- Sampling
- Results: e.g. band width limitation

Fourier – Why

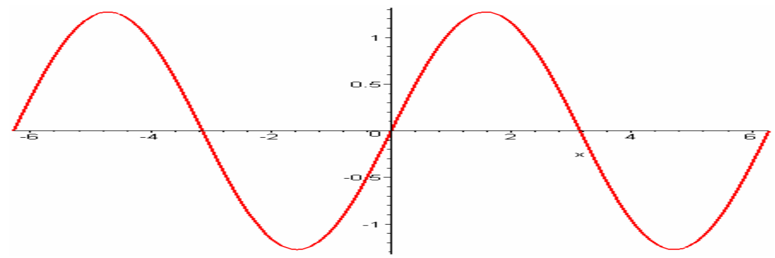
- 2D images are continous 2D functions: f(x, y)
- On screen, these images are represented by discrete samples, the pixels
- Sampling can cause artifacts
- Fourier space is **the** space for **analyzing and understanding** what happens when we sample.

Fourier Transform - Sum of sin and cos

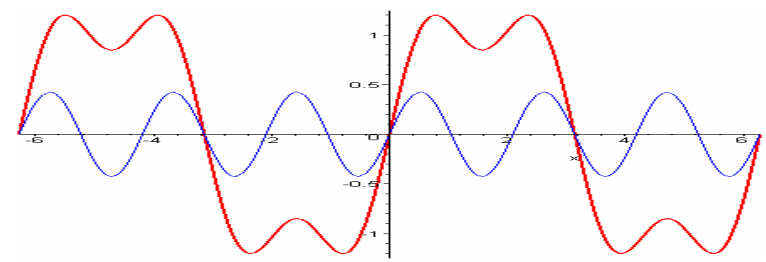
- *It can be shown that each periodic function can be represented by a sum of sin- and cos-functions, if the periods of the function statisfy certain simple conditions (Dirichlet's conditions about finitness of the periods).*
- *Like a projection of a vector onto basis vectors, a function gets projected onto basis functions => basis functions are orthogonal*

Sum of sin and cos – Example

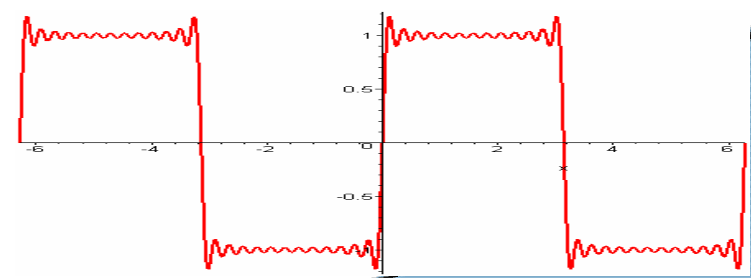
$$f(x)=\frac{4k}{\pi}\sin x$$



$$f(x)=\frac{4k}{\pi}\left(\sin x+\frac{1}{3}\sin 3x\right)$$



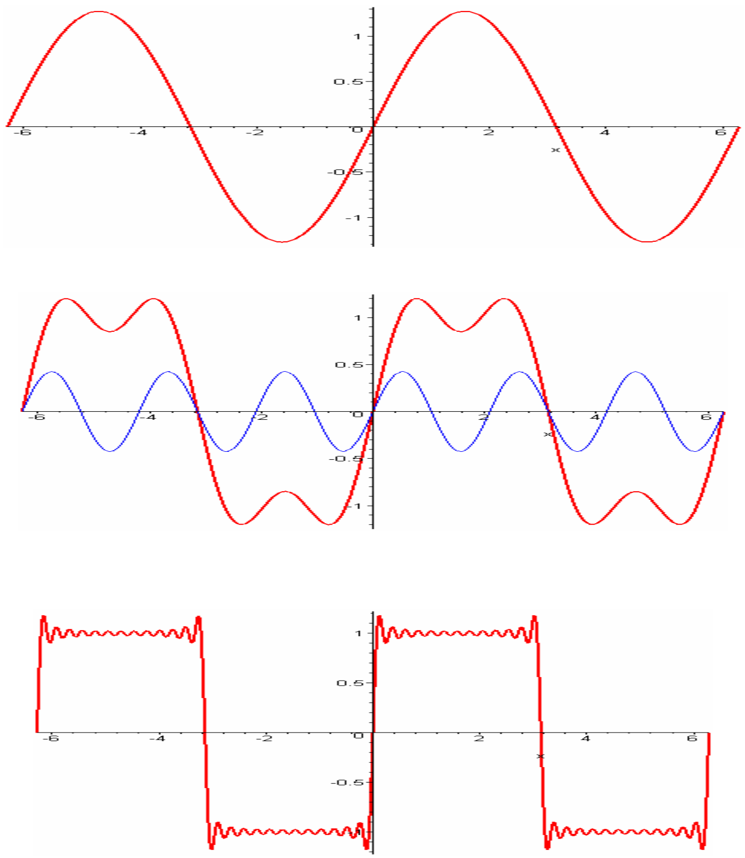
$$f(x)=\frac{4k}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\cdots+\frac{1}{25}\sin 25x\right)$$



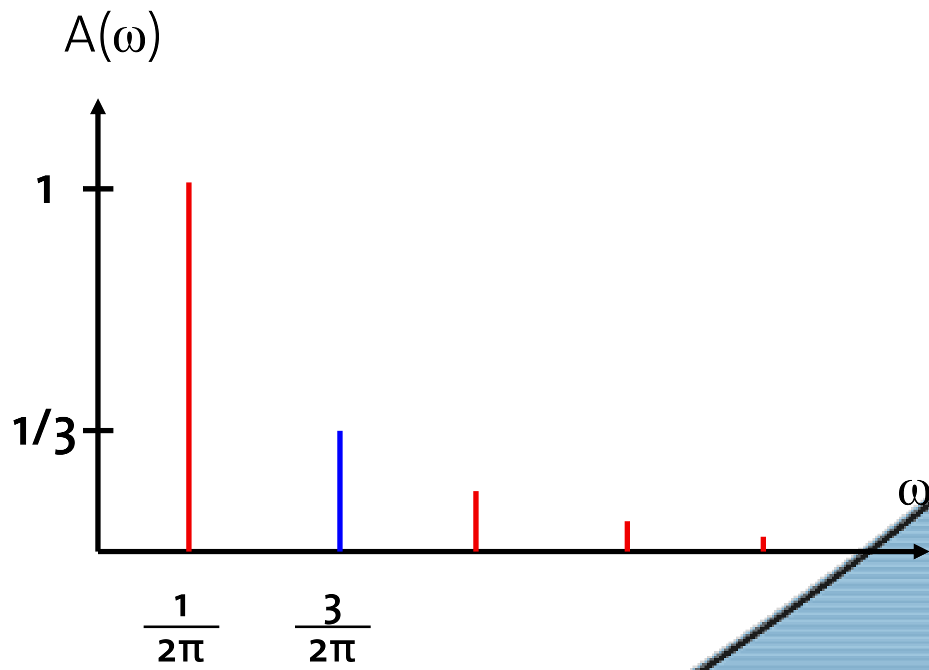
T = 2π   T = 2/3π   T = 2/25π

Spatial and Frequency Domain

Spatial



Frequency



Fourier Basics

$$F\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi ux}\, dx$$

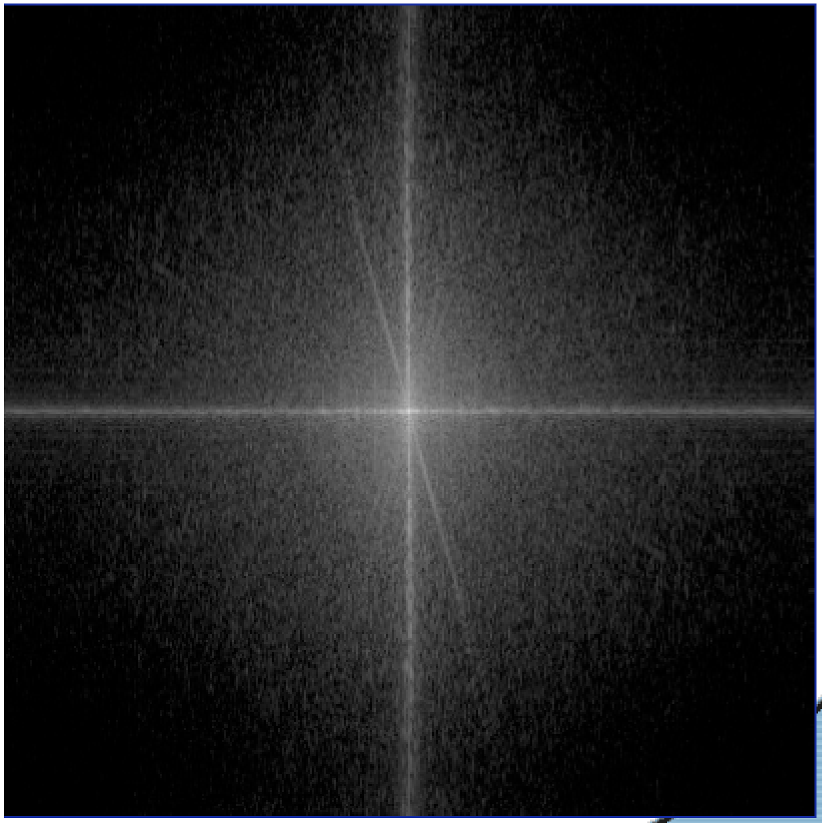
inverse: $$F^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u)\, e^{j2\pi ux}\, du$$

Euler: $$e^{jx} = cox(x) + j\sin(x)$$

Fourier – what else

- Rarely ever want to transform to Fourier space.
- By looking at F(u), we get a feel for the "frequencies" of the image.
- Space of Fourier transformed image: so-called frequency space.
- Intuitively: the sharper an edge, the higher the frequencies.

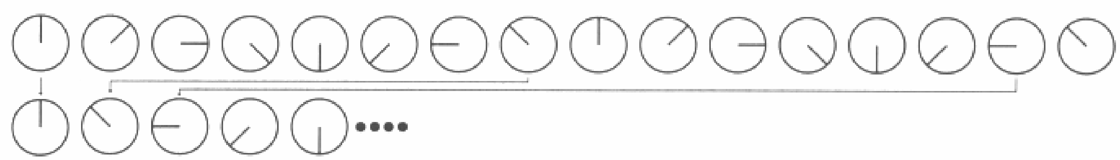
Spectrum of a 2D Image

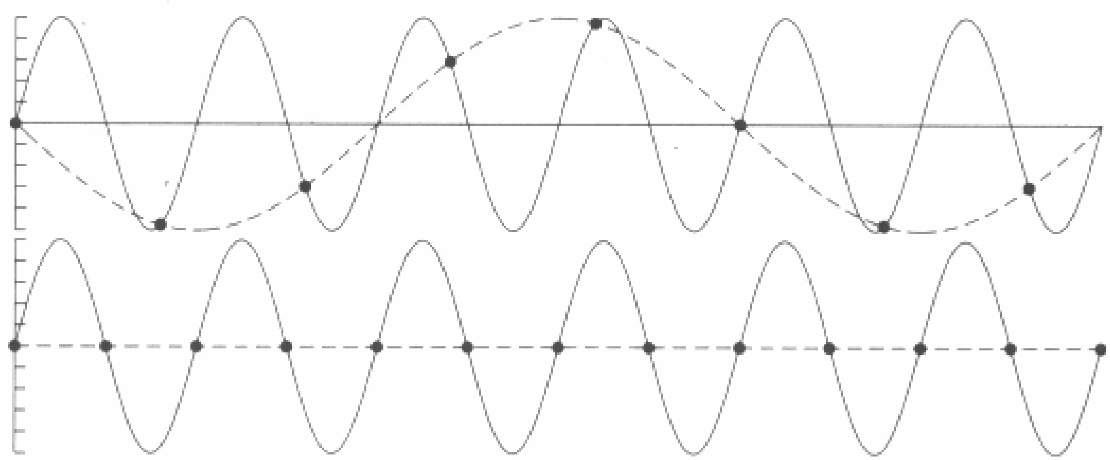


Source: Peter R. Fornaro

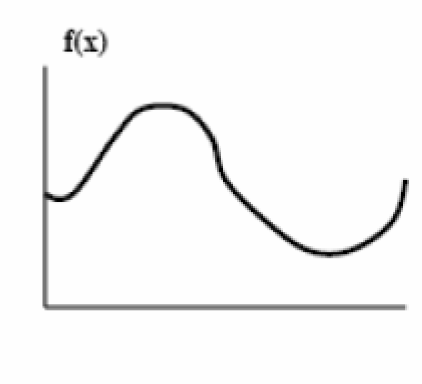
Sampling - Examples

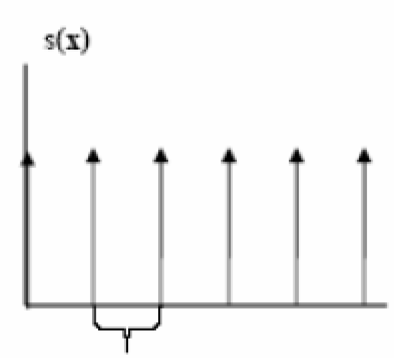
- Inadequate sampling

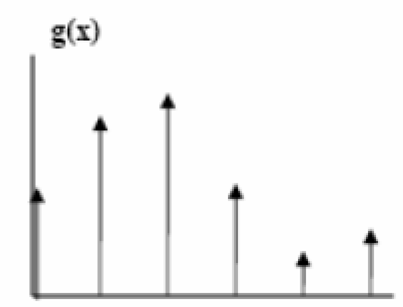




Sampling - Mathematically







Original function

- Sampling function
- Shah function
- Impuls train

-> delta fct.

f(x)s(x)

! f(x)*s(x)

Result of Analysis

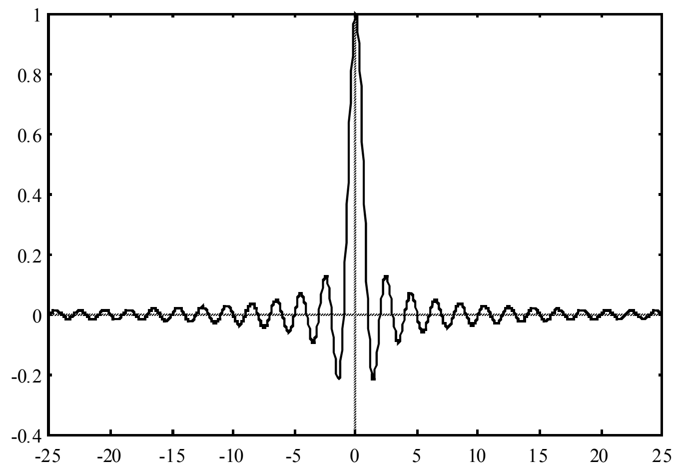
- „Fourier analysis for understanding aliasing“
- Selection of sampling rate (lecture):
  - Let W be the largest u for which |F(u)| > 0
  - Choose $u_{sampling} > 2W$ (Shannon)
- Problem: image has high frequencies (e.g. sharp edges), such that W too large or inexistent
  -> filter the signal

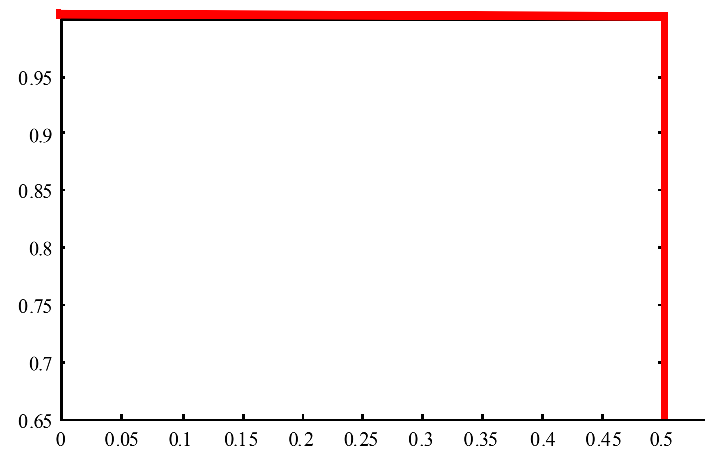
Filtering

Spatial

Frequency

Filter here: multiply with box filter





Cut-off high frequencies

Convolution Theorem

- Convolution theorem:
  convolution in the spatial domain is equivalent to multiplication in the frequency domain, and vica versa

$$f(x) * g(x) \equiv F(u)G(u)$$

$$f(x)g(x) \equiv F(u) * G(u)$$

- => image gets **convoluted** with sinc function
  => sinc function „visible“ in filtered image

Aufgabe 2 & 3

- Aufgabe 2:
  - a) describe more formally than your „friend“
  - b – e) ponder slides and script
- Aufgabe 3 a), b):

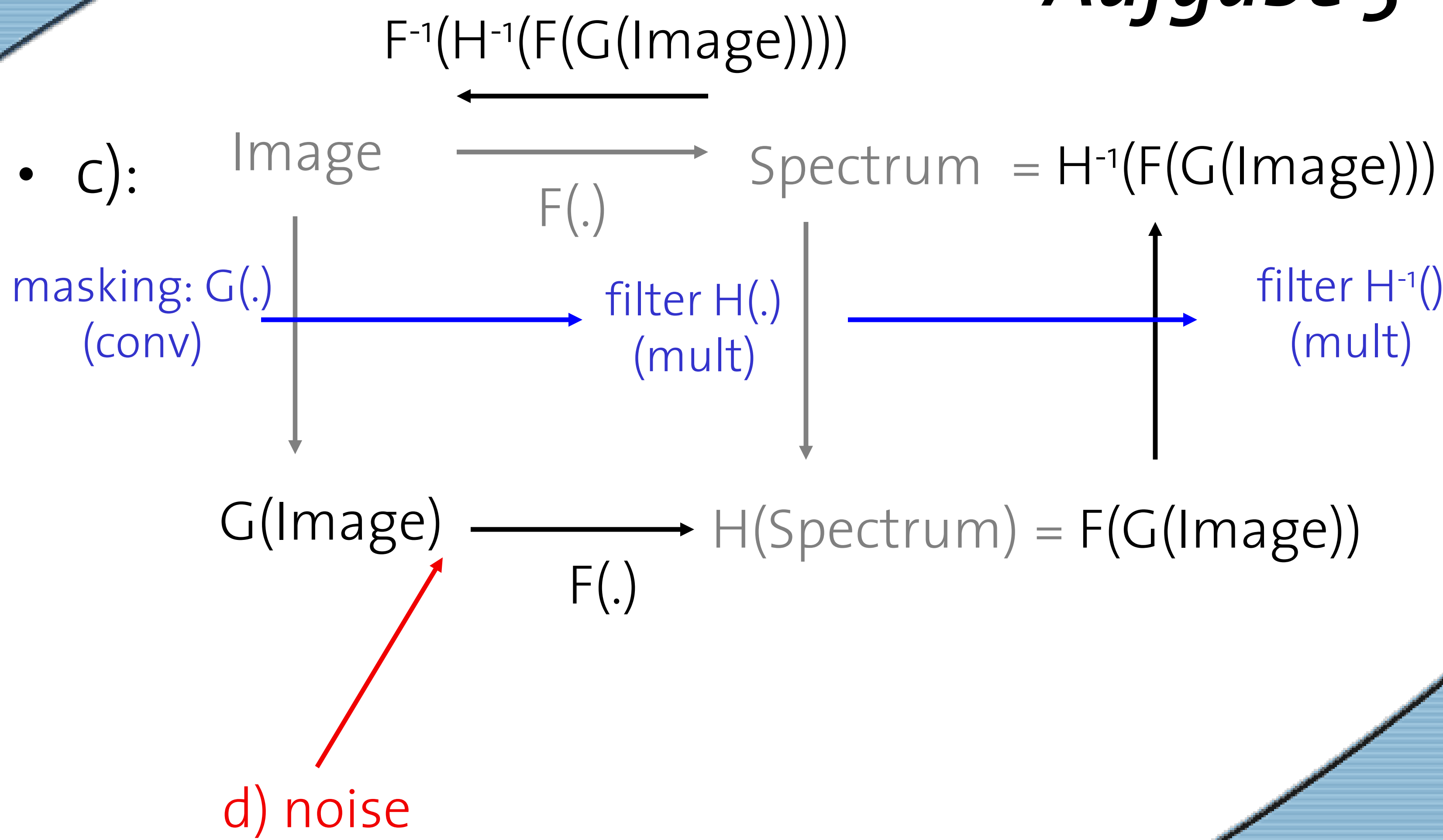
Image

↓ masking G (conv)

G(Image)

Aufgabe 3

- c):



F-1(H-1(F(G(Image))))
Image
F(.)
Spectrum = H-1(F(G(Image)))
masking: G(.) (conv)
filter H(.) (mult)
filter H-1() (mult)
G(Image)
F(.)
H(Spectrum) = F(G(Image))
d) noise

Aufgabe 4

- Applet
- Question: condition
  - easy: script, slides
  - Verify in applet
  - change settings
- free to experiment!

Aufgabe 5 - Reconstruction

- so far: sampling

Fourier transform of a sample fct. is again a sample fct.

f(x)

F(u)

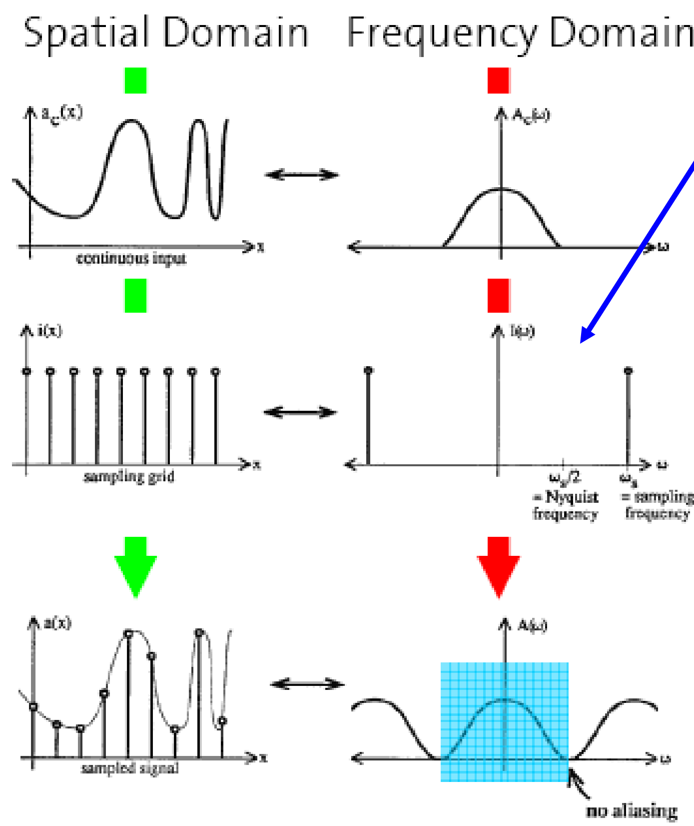
s(x)

S(u)

Aufgabe 1

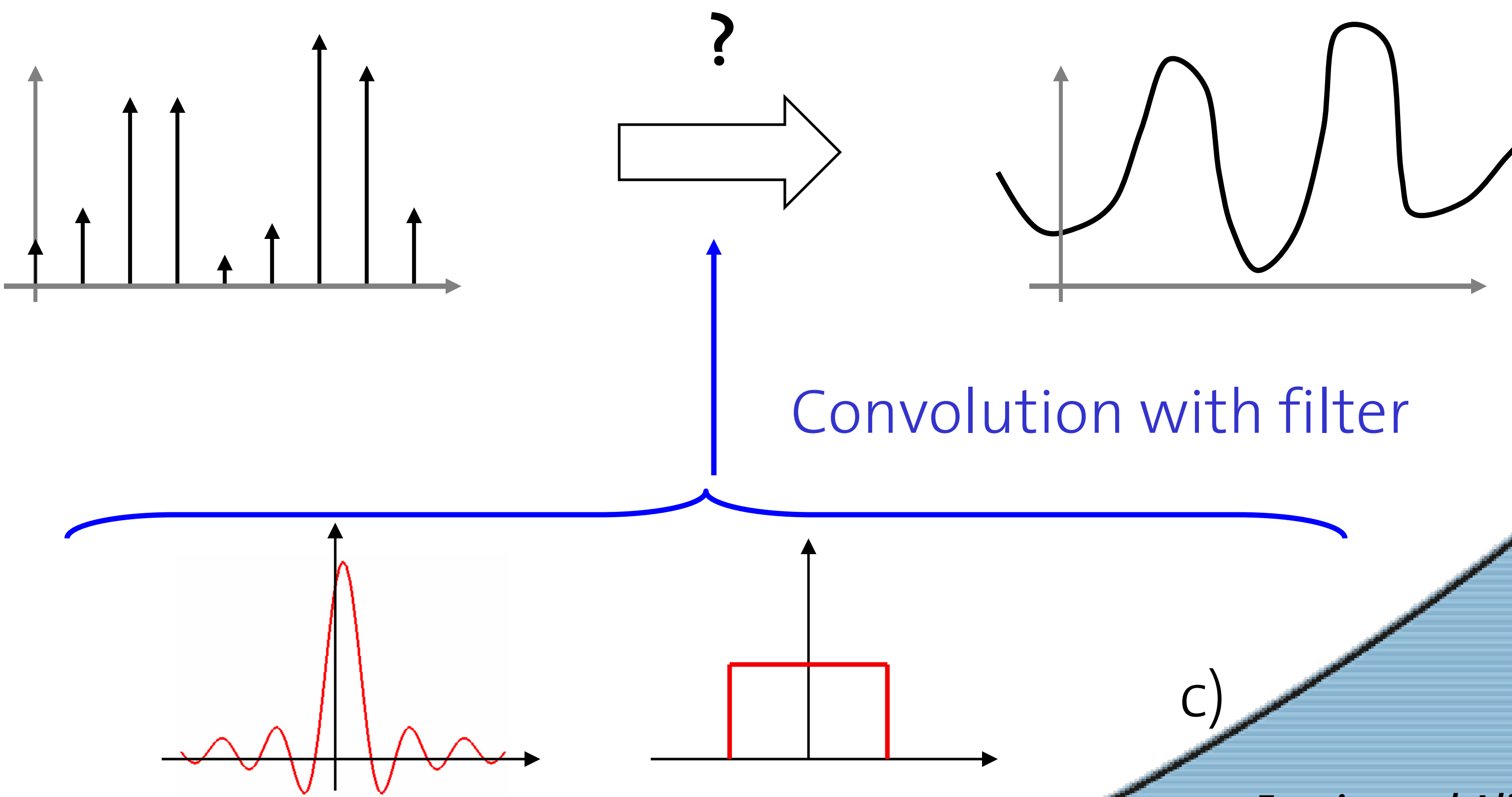
f(x)s(x)

F(u)*S(u)



Reconstruction

- Question:



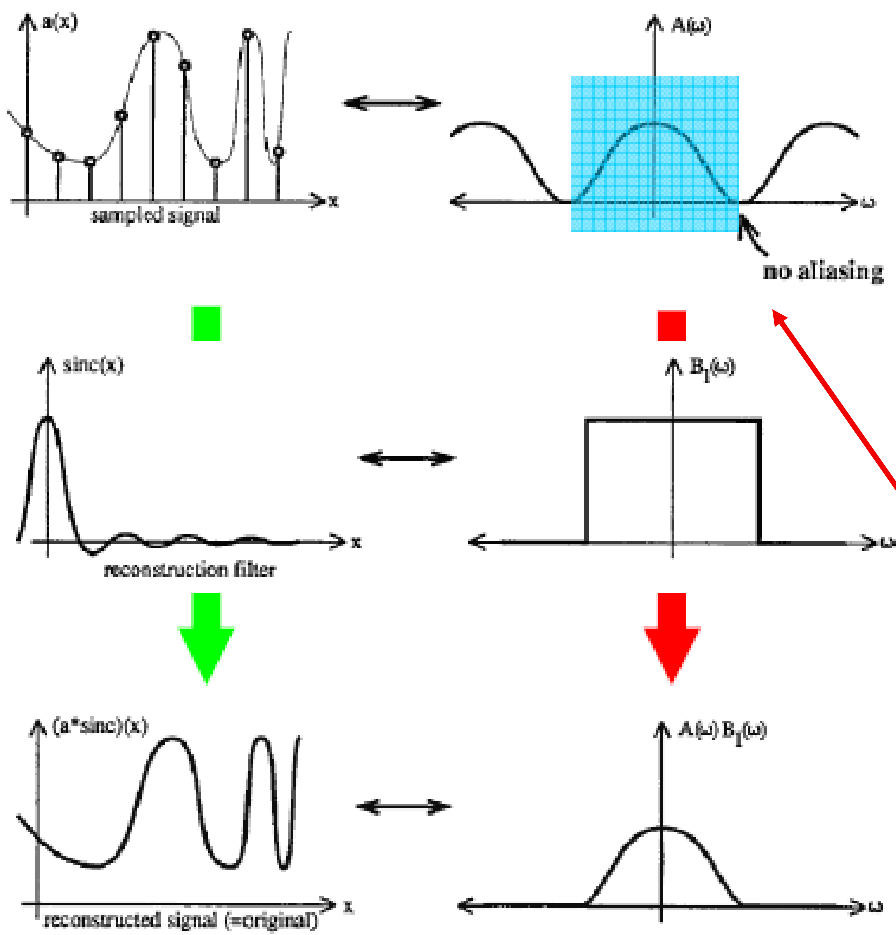
Reconstruction Problems

Spatial Domain Frequency Domain

f(x)s(x) F(u)*S(u)

filter FT filter

reconstructed



Explains problems
-> lecture

Aufgabe 5

- a) actually, the answer is in the lecture slides…
- b) convolution theorem and a)
- c)
  I. Aufgabe 1
  II. consider period of b)
  III. support