T4 – (Non-)linear Diffusion

Christian Vögeli
cvoegeli@inf.ethz.ch

Diffusion – Basics

- Idea:
  Use theory from phyiscs/chemistry for image processing.
- Diffusion: balancing of heat, chemical concentrations, fluids, material, …
  … and intensities in a picture
- As last exercise: we still search for (reconstruction) filters

Diffusion – Notation

- Gradient:
  - scalar function
  - „slope"

$$\nabla u(x,y) = \begin{pmatrix} \frac{\partial}{\partial x} u(x,y) \\ \frac{\partial}{\partial y} u(x,y) \end{pmatrix}$$

- Divergence:
  - vector field
  - „transpiration" material leaving a infintely small volume

$$\nabla \cdot j = div j = (\frac{\partial}{\partial x} j_x, \frac{\partial}{\partial y} j_y)$$

- Laplace operator:

$$\nabla^2 = \nabla \cdot \nabla = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$$

Diffusion – Laws

- **Fick's law**: concentration differences in u cause a flux j (vector field) compensating for the difference

$$j = -D \cdot \nabla u$$

material dependent diffusivity

- **Continuity equation:** changes in u over time can only be caused by moving material

$$\frac{\partial}{\partial t} u = -\nabla \cdot j$$

$$\nabla \cdot j = \lim_{V \to 0} \frac{\oint_S j \, da}{V}$$

Diffusion – Diffusion Equation

- relates temporal changes of u to the flux caused by u
- i.e. relates time and space

$$\frac{\partial}{\partial t} u = div(D \nabla u) = \nabla \cdot (D \nabla u)$$

- Image processing: „material" is intensity
  => D can depend on image structure

Diffusion – Diffusivity Defines Type

- Homogeneous diffusion: D does not depend on space
  - Ex: constant scalar diffusion Aufgabe 1
- Inhomogeneous diffusion: D is space dependent Aufgabe 2
- Nonlinear diffusion: D depends on u
- Isotropic diffusion: $\nabla u || j$
  - can be time and/or space dependent
  - Ex: scalar diffusion is linear isotropic

Aufgabe 1

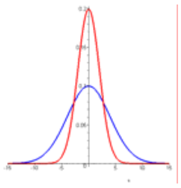
- isotropic: D = 1

$$\frac{\partial}{\partial t}u = \nabla \cdot (D\nabla u) = \nabla^2 u = \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u$$

D = 1   Laplacian

Aufgabe 1 a)

- diffused δ is Gaussian
- δ can be defined as a Gaussian:



$$\delta(x) = \lim_{\sigma \to 0} Gauss(x, \sigma)$$

Aufgabe 1 b), c), d)

- Numerically solve diffusion process:
  - Discretization
- b) Taylor expansion:

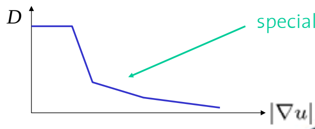
higher order terms dropped

$$u(t+\tau) \approx u(t) + \tau \frac{\partial}{\partial_t}u(t) \quad ....$$

  - Solve for derivative
  - Second derivative: recursive
  - ‚central differences': lecture slides/google
- c) plug in...
- d) analyze result from c)

Aufgabe 2

- Problem: linear isotropic diffusion blurs edges
- Solution: decrease the diffusion close to edges
  - Gradient is large at edges (see Laplacian of Gaussian in last exercise sheet)
  - Model D inverse proportional to $|\nabla u|$
- a): draw the desired behaviour in 1D



Aufgabe 2 b), c), d)

- Perona-Malik: special definition of D inverse proportional to $|\nabla u|$

$$D(|\nabla u|) = \frac{1}{1 + |\nabla u|^2/\lambda^2}$$

- b) 1D: divergence is simple derivative => substitute, solve.
- c) „step edge": non-ideal edge
- d) interpret, consult lecture slides

Aufgabe 3

- Yet another definition for

$$D(|\nabla u|)$$

- Matlab commands: solution last exercise
- a) self – explaining demos
- b) – d), f):
  - look at the jpg-image either as the faulty image – reconstruction
  - Simply treat diffusion as another filter
- e): actual reconstruction

Aufgabe 3

- b) cf. Aufgabe 1 a)
- c) λ „egde smooting and edge enhancement“
- d) keep λ fixed but increase time
- e): reconstruction
  - That‘s the actual idea of diffusion
  - Hint: will not result in the jpg-image
- f): cf. Aufgabe 1 d)